

Statistical Properties of E(5) and X(5) Symmetries

Jing Shu¹, Hong-bo Jia¹, Yu-xin Liu^{1,2,3,4,*}

¹ Department of Physics, Peking University, Beijing 100871, China

² The Key Laboratory of Heavy Ion Physics of the Chinese Ministry of Education,
Peking University, Beijing 100871, China

³ Institute of Theoretical Physics, Academia Sinica, Beijing 100080, China

⁴ Center of Theoretical Nuclear Physics, National Laboratory of
Heavy Ion Accelerator, Lanzhou 730000, China

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Abstract

We study the energy level statistics of the states in E(5) and X(5) dynamical symmetries. The calculated results indicate that the statistics of E(5) symmetry is regular and follows Poisson statistics, while that of X(5) symmetry involves two maxima in the nearest neighbor level spacing distribution $P(s)$ and the Δ_3 statistics follows the GOE statistics. It provides an evidence that the X(5) symmetry is at the critical point exhibiting competing degrees of freedom.

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*Corresponding author

The nature of “shape” phase transitions in finite quantal system is a fundamental issue and has been the subject of many investigations. In the last years, there have been many preeminent works concerning this subject in nucleus, including both theoretical calculations[1, 2, 3, 4] and experimental results[5, 6]. Particularly, two symmetries at the critical points of shape phase transition, called E(5) and X(5), have recently been proposed[7, 8]. It is a major breakthrough in the study of critical point behavior of nucleus undergoing a shape phase transition because it gives us simple analytic solutions while historically we have to resort to numerical calculations to describe such nucleus.

Empirical examples of such kind of symmetries have been found soon after they were put forward. For instances, the ^{134}Ba , ^{102}Pd , ^{108}Pd , Ru isotopes are proposed to be the ones in E(5) symmetry[9, 10, 11, 12] and the ^{152}Sm , ^{150}Nd , ^{104}Mo in X(5) symmetry[13, 14, 15]. These examples show that the obtained new symmetries represent a helpful theoretical tool which could well describe the structure of the realistic nucleus at the critical point of a shape transition, in a sense complementing the role played by the dynamical symmetries of the Interacting Boson Model(IBM)[16, 17]. However, there remain some interesting aspects on the statistical properties of the E(5) and X(5) symmetries. (1) Whether the nuclear system displays ordered or chaotic spectrum in the new symmetries? (2) Does the statistics give a strong evidence to support that the obtained new symmetries can well describe the nucleus at the critical point in a shape transition? (3) The new symmetries are based on the particular potential amenable to analytic descriptions which well approximate to the “true” potential. Does the simplification of the potential eliminates part of the statistical properties of the realistic nucleus at the critical point? Or more concretely, are the results consistent with the statistical properties obtained by Alhassid and collaborators[18, 19, 20] in the framework of IBM? Therefore it is crucially important to obtain the statistics of the new symmetries and answer the above

opening questions. In this sense, we explore the statistical properties of E(5) and X(5) symmetries in the paper.

We start from the Bohr Hamiltonian[21]

$$H = -\frac{\hbar^2}{2B} \left[\frac{1}{\beta^4} \frac{\partial}{\partial \beta} \beta^4 \frac{\partial}{\partial \beta} + \frac{1}{\beta^2 \sin 3\gamma} \frac{\partial}{\partial \gamma} \sin 3\gamma \frac{\partial}{\partial \gamma} - \frac{1}{4\beta^2} \sum_{\kappa} \frac{Q_{\kappa}^2}{\sin^2(\gamma - \frac{2}{3}\pi\kappa)} \right] + V(\beta, \gamma). \quad (1)$$

In the E(5) symmetry, the potential $V(\beta, \gamma)$ depends only on the deformation parameter β and it can be written as $V(\beta, \gamma) = U(\beta)$. Considering the case, in which the potential is simplified as an infinite well, one has the eigenvalues[7]

$$E_{\xi, \tau} = \frac{\hbar^2}{2B} \left(\frac{\chi_{\xi, \tau}}{\beta_W} \right)^2, \quad (2)$$

where B is a parameter, $\chi_{\xi, \tau}$ is the ξ th zero of the Bessel function $J_{\nu}(z)$, and the order of the Bessel function is $\nu = \tau + \frac{3}{2}$ with τ being the quantum number similar to the irreducible representation of O(5) group in the IBM, β_W is the width of the well.

In the X(5) symmetry, the potential was supposed to be a square well in the variable β and a harmonic oscillator in γ with no $\beta - \gamma$ couplings, which is a good approximation in the U(5)-SU(3) shape transition. In this case, the eigenvalues are given as[8]

$$E(s, L, n_{\gamma}, K, M) = E_{\beta} + E_{\gamma}, \quad (3)$$

where

$$E_{\beta} = \left(\frac{\chi_{s, L}}{\beta_W} \right)^2 \quad (4)$$

and

$$E_{\gamma} = \frac{3a}{\sqrt{\langle \beta^2 \rangle}} (n_{\gamma} + 1) - \frac{4}{3} \frac{(K/2)^2}{\langle \beta^2 \rangle}. \quad (5)$$

The $\chi_{s, L}$ is the s th zero of Bessel function $J_{\nu}(z)$ of (irrational) order $\nu = [J(J+1)/3 + 9/4]^{1/2}$, L is the total angular momentum (with projections K on the symmetry axis and M on the quantization axis in the laboratory frame). a is a parameter related to the

interaction strength of the potential $U_\gamma = (3a)^2\gamma^2/2$ and n_γ is the number of γ -vibration quanta, while β_w is the same as that in E(5) symmetry and the $\langle\beta^2\rangle$ is the average of β^2 over the wave functions. Combining all variables, one obtains the most general expression[8]

$$E(s, L, n_\gamma, K, M) = E_0 + B(\chi_{s,L})^2 + An_\gamma + CK^2, \quad (6)$$

where

$$A = \frac{3a}{\sqrt{\langle\beta^2\rangle}} \quad (7)$$

and

$$B = \left(\frac{1}{\beta_w}\right)^2. \quad (8)$$

Before analyzing the energy level statistics, we estimate the relative value of the parameters A/B in the light of the “true” potential given in the Fig.1. of Ref.[8]. We can see from the figure that the width of the infinite well to approximate the “true” potential is $\beta_w \approx 0.8$, and most of the quantum states can thus be restricted in $U_\beta < 0.5$ (arbitrary unit). If we assume that the total energy of the quantum states are conserved in $E = 0.5$, the quantum states of the γ degree are then mainly restricted in $U_\gamma < 0.5$, as the potential is completely decoupled into two separate parts in β and γ respectively. Since the X(5) symmetry is obtained around $\gamma = 0$ [8], the corresponding restriction in γ by the total energy must be very small. Taking this small value of γ as 0.1, we then work out that the strength of the two dimensional oscillator ($3a$) must be more than 10 according to the formulation $U_\gamma = (3a)^2\gamma^2/2$. With Eqs.(7) and (8), we obtain that the parameter A must be more than 25 if we take $\sqrt{\langle\beta^2\rangle}$ as 0.4 and the parameter $B = 1.56$. Consequently we get the relative value $A/B \geq 16$.

Our estimate is consistent with the recent empirical example ^{104}Mo [15] of X(5) symmetry, where the nucleus presents the X(5) pattern not only in its ground state band, but also in its low-lying $n_\gamma = 1, 2; K = 2n_\gamma$ bands. From the level scheme given in the Fig.2.

of Ref.[15], we can obtain the empirical values of the parameters, $A \approx 800$ and $B \approx 30$ (in keV). The relative value is clearly in the region of $A/B \geq 16$ as we have proposed. Such a consistence supports that the assumption to obtain X(5) symmetry around $\gamma = 0$ in the Ref.[8] is quite reasonable since in fact, most of the quantum states in the realistic nucleus at the critical point of the transition from spherical to axially-deformed shape are restricted in $\gamma < 0.1$.

To analyze the energy level statistics in the dynamical symmetries, we take the following process. At first with Eqs.(2) and (6), we calculate the energy levels of the nucleus in E(5) and X(5) symmetries, respectively. The parameter ξ and τ in E(5) symmetry are truncated manually at a maximal number $\xi^{(m)}$ and $\tau^{(m)}$. We also truncated s and n_γ at $s^{(m)}$ and $n_\gamma^{(m)}$ respectively in X(5) symmetry. After the spectrum $\{E_i\}$ has been determined, it is necessary to separate the smoothed average part whose behavior is nonuniversal and can not be described by random-matrix theory(RMT)[22]. To do so we count the number of the levels below E so that one can define a staircase function $N(E)$ of the spectrum (see for example Ref.[18])

$$N(E) = N_{av}(E) + N_{fluct}(E). \quad (10)$$

Then we fix the $N_{av}(E_i)$ semiclassically by taking a smooth polynomial function to fit the staircase function $N(E)$. We obtain finally the unfolded spectrum with the mapping

$$\{\tilde{E}_i\} = N(E_i). \quad (11)$$

This unfolded level sequence $\{\tilde{E}_i\}$ is obviously dimensionless and has a constant average spacing of 1, but the actual spacings exhibit frequently strong fluctuation.

We have used two statistical measures to determine the fluctuation properties of the unfolded levels: the nearest neighbor level spacings distribution(NSD) $P(S)$ and the spectral rigidity $\Delta_3(L)$. The nearest neighbor level spacing is defined as $S_i = (\tilde{E}_{i+1}) - (\tilde{E}_i)$.

The distribution $P(S)$ is defined as that $P(S)dS$ is the probability for the S_i to lie within the infinitesimal interval $[S, S + dS]$. For a regular system, it is expected to behave like the Poisson statistics

$$P(S) = e^{-S}, \quad (12)$$

whereas if the system is chaotic, one expects to obtain the Wigner distribution

$$P(S) = (\pi/2)S \exp(-\pi S^2/4), \quad (13)$$

which is consistent with the GOE statistics[22, 24, 23]. With the Brody parameter ω in the Brody distribution

$$P_\omega(S) = \alpha(1 + \omega)S^\omega \exp(-\alpha S^{(1+\omega)}), \quad (14)$$

where

$$\alpha = \Gamma[(2 + \omega)/(1 + \omega)]^{1/2} \quad (15)$$

and $\Gamma[x]$ is the Γ function, the transition from regularity to chaos can be measured with the Brody parameter ω quantitatively. It is evident that $\omega = 1$ corresponds to the GOE distribution, while $\omega = 0$ to the Poisson distribution. A value $0 < \omega < 1$ means an interplay between the regular and the chaotic.

As to the spectral rigidity $\Delta_3(L)$, it is defined as

$$\Delta_3(L) = \left\langle \min_{A,B} \frac{1}{L} \int_{-L/2}^{L/2} [N(x) - Ax - B]^2 dx \right\rangle, \quad (16)$$

where $N(x)$ is the staircase function of a unfolded spectrum in the interval $[-L/2, x]$. The minimum is taken with respect to the parameters A and B . The average denoted by $\langle \cdots \rangle$ is taken over a suitable energy interval over x . Thus from this definition $\Delta_3(L)$ is the local average least square deviation of the staircase function $N(x)$ from the best

fitting straight line. For the GOE the expected value of $\Delta_3(L)$ can only be evaluated numerically, but it approaches the value

$$\Delta_3(L) \cong (\ln L - 0.0687)/\pi^2 \quad (17)$$

for large L . And for Poisson statistics

$$\Delta_3(L) = L/15. \quad (18)$$

In our calculation of E(5) symmetry, all the degenerate states caused by additional quantum number are considered just as one single state[25]. That is because we manually introduced the additional quantum number to distinguish the degenerate states with the same quantum number τ . The degenerate states with different additional quantum numbers are statistically uncorrelated since no interaction is involved to link such states. If we do not remove the degeneracy, we would take the mixed ensembles (with different additional quantum numbers which are good quantum numbers in the dynamical symmetries) into consideration and obtain the over-Poisson distribution in practical calculations since nearly 1/4 of the spacings of all are zero. In order to evaluate the fluctuations of the energy levels in one pure ensemble, these large amount of the statistically uncorrelated states should be removed.

At first, we analyze the energy levels given in Eqs.(2) for the E(5) dynamical symmetry. The numerical results of the NSD and Δ_3 statistics show that the statistical feature does not depend on the angular momentum obviously. We then illustrate only those of the states with low spin-parity $J^\pi = 0^+$ but different sets of truncated numbers $\xi^{(m)}$ and $\tau^{(m)}$ of E(5) symmetry in Fig.1 (a), (b) and (c). Meanwhile the Brody parameter ω of the NSD[24] is also evaluated and listed in the figures. The numerical results show that the statistics of the E(5) symmetry is quite close to Poisson statistics. When the number of the energy levels involved in the statistics (denoted by $\xi^{(m)}$ and $\tau^{(m)}$) increases, the

statistics becomes almost exactly the Poisson-type. It indicates that the E(5) symmetry is a dynamical symmetry which is completely integrable in classical limit.

We then analyze the statistics in the X(5) symmetry. We can easily find that the energy eigenvalues in the X(5) symmetry contain two terms, one is the square of the zeros of the Bessel function in β , another is the solution of two dimensional oscillator function in γ . In order to display how the relative strength of the two terms affects the energy level statistics, we calculate the statistics in different values of the relative strength A/B , the results are displayed in Fig.2. All of the relative strength A/B satisfy the condition $A/B \geq 16$ as we estimated above. It is apparent that when the parameter A/B is not very large, two maxima[25] appear in the NSD statistics (see the left part of Fig.2. (a), (b)), which is rather different from the fluctuation properties in other cases except for those in the U(5) dynamical symmetry with collective backbending[25, 26]. In theoretical point of view, the X(5) symmetry describes the critical point nuclei in a spherical to axially deformed shape phase transition analytically. The structure of its energy levels must be not the same everywhere. In fact, the term $B(\chi_{s,L})^2$ is approximately a rotational term of s , whereas the term An_γ is a vibrational term of n_γ . The simultaneous appearance of these two terms induces a competition and makes the ensemble with the same angular momentum L and its projection on the symmetry axis K involves two sequences, one of which is in vibration, another is in rotation. When we unfold the above spectrum $\{E_i(L)\}$, the two sequences are normalized with an unique total average spacing, and their maxima in the spacing distributions are no longer the same, consequently two maxima emerge because of the uniform unfolding procedure. The same mechanism and results can be found in U(5) dynamical symmetry with collective backbending, where the structure of the yrast band changes from the U(5) vibrational states to the rotational states with full d-boson number $n_d = N$, which implies that a transition of collective motion mode

may happen as the angular momentum increases. In this statistical point of view, the X(5) symmetry describes well the critical point behavior of nucleus undergoing a shape transition from U(5) dynamical symmetry to SU(3) dynamical symmetry which involves competing degrees of freedom.

In the Δ_3 statistics when A is not very large (see the right part of Fig.2. (a), (b)), it is clearly that the statistics follows the GOE statistics, which suggests that the competing strengthes in the β and γ degrees can cause the onset of chaos in X(5) symmetry. The case that the degree of chaoticity reaches its maximal when different competing strengthes coexist and are comparable can be also found in other works (see for example Ref.[25, 27]).

Looking over Fig.2.(c) and (d), one can recognize that when we increase the strength of the oscillator in γ degree of freedom, which corresponds to the parameter A , the two maxima in the NSD statistics soon vanishes and both the NSD and Δ_3 statistics become close to the Poisson statistics. It indicates that only the vibrational mode plays a dominant role in the system and the competition mentioned above does not exist. If we go to extremes to the case when the vibrational strength $3a \rightarrow \infty$, the γ degree of freedom no longer exists. The X(5) symmetry becomes the E(5) symmetry in a four-dimensional space, and its statistics is certainly regular.

The results obtained above are consistent with Alhassid and collaborates' work[19]. The E(5) symmetry lies in the transitional region between a vibration to a γ -unstable rotation, which is regular both classically and quantum mechanically in the work. It has been known that the Hamiltonian in the consistent Q formalism[28] of the IBM can be given as

$$H = \epsilon \hat{n}_d - \kappa \hat{Q}^2(\chi) \cdot \hat{Q}^2(\chi), \quad (9)$$

where

$$\hat{Q}_\mu^2(\chi) = (s^\dagger \tilde{d} + d^\dagger s)_\mu^2 + \chi (d^\dagger \tilde{d})_\mu^2.$$

By changing ϵ/κ under restriction $\chi = -\sqrt{7}/2$, one can describe the U(5)-SU(3) transition with this Hamiltonian. To discuss the shape phase transition and relate it with X(5) symmetry, we reparametrize the ration ϵ/κ as $(1 - \zeta)/\zeta$. From the Ref.[1] we know that the flat potential near the critical point could be obtained when the parameter $\zeta \in (0.025, 0.029)$ [1] in case that the boson number $N = 10$. We then transfer the parameter ζ into parameter η used in Alhassid and collaborates' work[19]. The relation is

$$\eta = \frac{1 - \zeta}{1 - \zeta + N\zeta},$$

and the value of η near the critical point is $0.770 \sim 0.796$. It is clearly that the critical point which corresponds to the X(5) symmetry lies in the most chaotic region(see the Fig.1. in Ref[19]).

Finally, we turn to the question we have raised in the beginning. In X(5) symmetry the nucleus exhibits chaotic spectrum, which is probably caused by the comparable competing strengthes in the β and γ degrees. The two maxima appearing in the NSD statistics suggest that the spectrum splits into two different sequences, one is rotational and the other is vibrational. Such a behavior illustrates that the nucleus in X(5) symmetry is at the critical point of the transition from spherical to axially-deformed shape with competing degrees of freedom. While the statistics in E(5) symmetry provides no evidence for such a drastic competition in the critical shape. That might be because the E(5) symmetry is a critical symmetry corresponding to an isolated point of second order phase transition. Very recently, it has been pointed out that such a second-order phase transition located at the triple-point of the three phases[29]. The shape coexistence region shrinks[1] as the parameter χ in the Casten's triangle changes from $-\sqrt{7}/2$ to 0. In deed, as a general rule, phase coexistence can be found only in first-order phase transitions[30]. There is no shape coexistence in E(5) symmetry and the spectrum does not split into two sequences, then no competition exists and its statistics is uniform(no two maxima in the NSD statistics).

The common subalgebra $O(5)$ in the $U(5)$ – $O(6)$ transition makes the statistics regular in this region and the ordered spectrum of nucleus in $E(5)$ symmetry confirms the past work[19]. It is worth to mention that the bifurcation effect(number of maxima more than one) in the NSD statistics might be a useful tool to determine whether the system is at the critical point of a first order shape transition or shape coexistence which involves competing degrees of freedom.

In summary, we have calculated the statistics of $E(5)$ and $X(5)$ symmetries. The statistics of $X(5)$ symmetry involves two maxima in the NSD statistics and the Δ_3 statistics follows the GOE statistics, which provides an evidence that the $X(5)$ symmetry is at the critical point exhibits competing degrees of freedom. The statistics of $E(5)$ symmetry is regular. That is probably caused by the common subgroup $O(5)$ in the $U(5)$ – $O(6)$ transition. Since the $E(5)$ symmetry corresponds to the second order shape phase transition with no shape coexistence, then no such competition exists as in the $X(5)$ symmetry.

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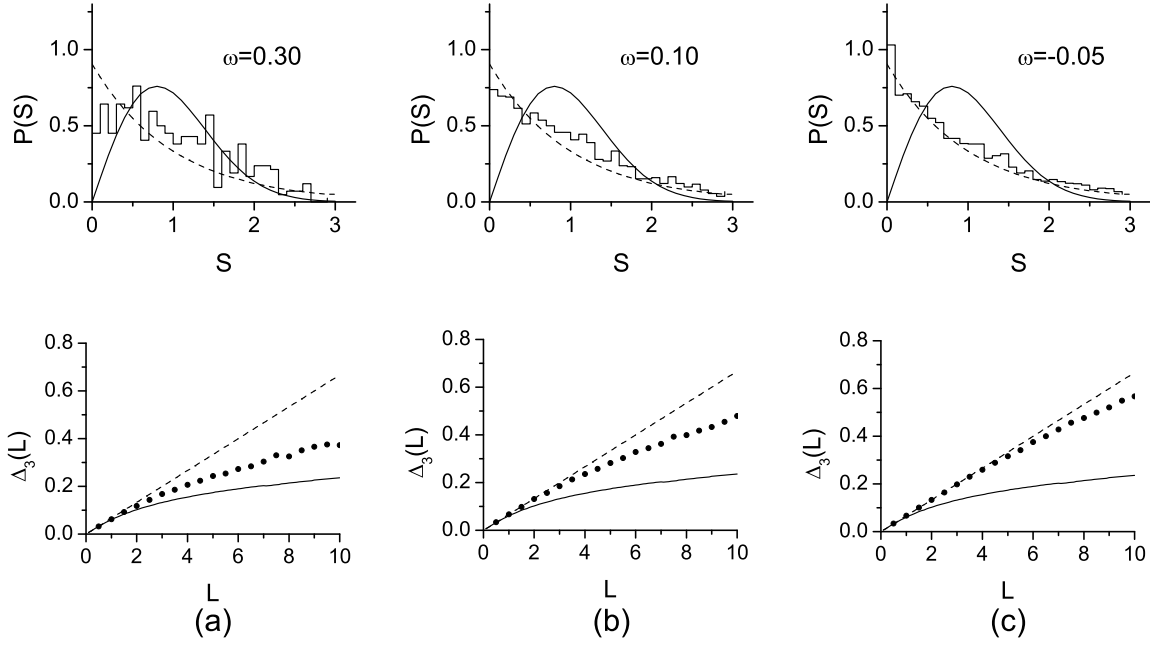


Figure 1: Comparison of energy level statistics of the states $J^\pi = 0^+$ in E(5) symmetry with different numbers of energy levels: (a) for $\tau^{(m)} = 20$, $\xi^{(m)} = 20$, (b) for $\tau^{(m)} = 40$, $\xi^{(m)} = 40$, and (c) for $\tau^{(m)} = 80$, $\xi^{(m)} = 80$. In all figures, the solid lines and dashed lines describe the GOE and Poisson statistics, respectively.

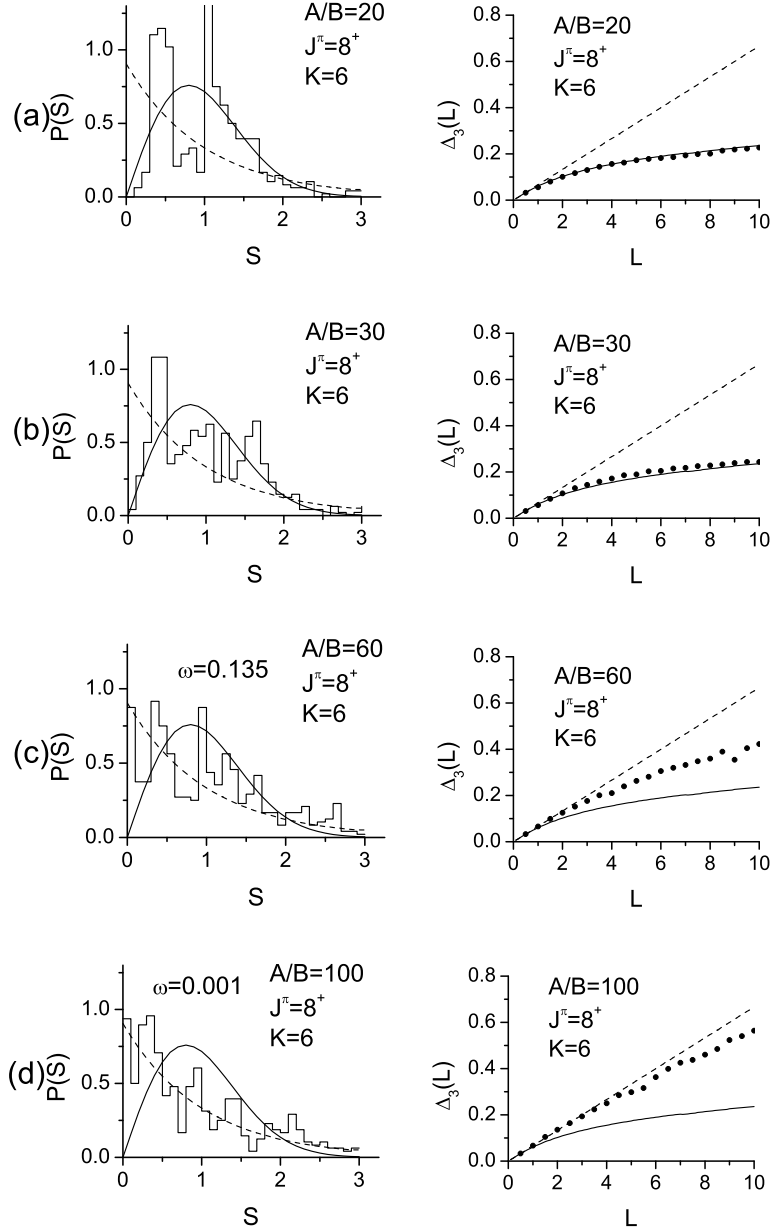


Figure 2: Comparison of energy level statistics of the states $J^\pi = 8^+$ and $K = 6$ in $X(5)$ symmetry with the same manually truncated quantum number $s^{(m)} = 30$ and $n_\gamma^{(m)} = 30$, but different relative strength A/B : (a) for $A/B = 20$, (b) for $A/B = 30$, (c) for $A/B = 60$, and (d) for $A/B = 100$.